A completeness relation for the coherent states of the (p,q)-oscillator by (p,q)-integration

1994 J. Phys. A: Math. Gen. 27 L375

(http://iopscience.iop.org/0305-4470/27/11/005)

J. Phys. A: Math. Gen. 27 (1994) L375–L379. Printed in the UK

LETTER TO THE EDITOR

A completeness relation for the coherent states of the (p, q)-oscillator by (p, q)-integration

Gong Ren-Shan

CCAST (World Laboratory), PO Box 8730, Beijing 100080, Peoples' Republic of China and (mailing address) Department of Physics, Nanchang University, Jiangxi 330047, People's Republic of China

Received 8 April 1994

Abstract. (p,q)-integration is defined for the (p,q)-oscillator. This is used to prove a completeness relation for the coherent states of the (p,q)-oscillator. The (p,q)-analogue of the Bargmann–Fock representation is also discussed.

In recent years, the quantum algebras with multiparameter deformations have aroused much interest [1–7] because they allow for more flexibility when dealing with applications to concrete physical models. In [4], the (p,q)-oscillator and their coherent states had been proposed for obtaining the realization of two-parameter quantum algebras. The major object of this letter is to derive a completeness relation for the (p,q)-oscillator coherent states. First, we have defined a (p,q)-integration and derived the (p,q)-analogue of Euler's formula for $\Gamma(x)$, on which the completeness relation of the (p,q)-oscillator coherent states is based. Second, we propose a proper integration measure and prove that there exists a resolution of unity for the (p,q)-oscillator coherent states. Finally, the (p,q)-analogue of the Bargmann–Fock representation is simply discussed.

Let us first review some useful results previously obtained for the (p,q)-oscillator. The latter is defined in terms of a (p,q)-creation operator $A^\dagger$, a (p,q)-annihilation operator $A=(A^\dagger)^\dagger$ and a Hermitian number operator N, satisfying the following relations [4]

$$[N,A^\dagger]=A^\dagger \qquad [N,A]=-A \tag{1a}$$

$$AA^\dagger - qA^\dagger A = p^{-N} \tag{1b}$$

$$AA^\dagger - p^{-1}A^\dagger A = q^N. \tag{1c}$$

The formulas (1b) and (1c) can also be written as

$$[A,A^\dagger]=[N+1]_{p,q}-[N]_{p,q} \tag{2}$$

where

$$[x]_{p,q}=\frac{q^x-p^{-x}}{q-p^{-1}}. \tag{3}$$

Note that there exists the $q \leftrightarrow p^{-1}$ symmetry in (1b), (1c) and (2), and in the limit $p=q$, (p,q)-oscillator reduces to q-oscillator [8–9]. Although the deformation parameters p and

0305-4470/94/110375+05$19.50

q may be varied independently, we take, throughout, $0 < q < 1$, $p > 0$ and such, that $[n]_{p,q} > 0$ for any $n > 0$, unless otherwise specified.

In the Fock space with $\{|n\rangle|(n = 0, 1, 2, \ldots)\}$ as the complete orthonormal set of eigenstates of N, one has [4]

$$A|n\rangle = \sqrt{[n]_{p,q}}|n-1\rangle \qquad A^{\dagger}|n\rangle = \sqrt{[n+1]_{p,q}}|n+1\rangle \qquad N|n\rangle = n|n\rangle \tag{4a}$$

$$|n\rangle = \{[n]_{p,q}!\}^{-1/2}(A^{\dagger})^n|0\rangle \qquad [n]_{p,q}! = \prod_{k=1}^{n}[k]_{p,q} \tag{4b}$$

where $\langle n|m\rangle = \delta_{n,m}$. The resolution of unity may be written as

$$\sum_{n=0}^{\infty}|n\rangle\langle n| = I. \tag{5}$$

The (p, q)-oscillator coherent states, defined by

$$A|z\rangle_{p,q} = z|z\rangle_{p,q} \tag{6}$$

can be written in terms of a (p, q)-exponential as follows

$$|z\rangle_{p,q} = \exp_{p,q}(zA^{\dagger})|0\rangle \equiv \sum_{n=0}^{\infty}\frac{(zA^{\dagger})^n}{[n]_{p,q}!}|0\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]_{p,q}!}}|n\rangle. \tag{7}$$

These coherent states are not to be orthonormal. In fact, we have

$${}_{p,q}\langle z'|z\rangle_{p,q} = \exp_{p,q}(\bar{z}'z). \tag{8}$$

According to [4], the (p, q)-derivative is defined to be

$$\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}\psi(x) = \frac{\psi(qx) - \psi(p^{-1}x)}{qx - p^{-1}x} \tag{9}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}(\alpha x^n) = \alpha[n]_{p,q}x^{n-1} \tag{10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}\exp_{p,q}(\alpha x) = \alpha\exp_{p,q}(\alpha x) \tag{10b}$$

where α is a constant. We now introduce the (p, q)-integration for $f(x)$ on the interval $[0, a]$ as

$$\int_0^a f(x)\,\mathrm{d}_{p,q}x = \begin{cases} (q - p^{-1})a\sum_{n=0}^{\infty}q^{-(n+1)}p^{-n}f(q^{-(n+1)}p^{-n}a) & \text{for } qp > 1 \\ (p^{-1} - q)a\sum_{n=0}^{\infty}p^{(n+1)}q^{n}f(p^{(n+1)}q^{n}a) & \text{for } qp < 1 \end{cases} \tag{11}$$

and for the interval $[0, \infty)$

$$\int_0^\infty f(x)\,\mathrm{d}_{p,q}x = \begin{cases} (q-p^{-1})\sum_{n=-\infty}^{\infty} q^{-(n+1)}p^{-n} f(q^{-(n+1)}p^{-n}) & \text{for } qp > 1 \\ (p^{-1}-q)\sum_{n=-\infty}^{\infty} p^{(n+1)}q^{n} f(p^{(n+1)}q^{n}) & \text{for } qp < 1 \end{cases} \tag{12}$$

It is obvious that the (p, q)-integration is the inverse operation of the (p, q)-derivative, so we have

$$\int \alpha x^{n-1}\,\mathrm{d}_{p,q}x = \frac{\alpha}{[n]_{p,q}} x^n + \text{const.} \tag{13a}$$

$$\int \exp_{p,q}(\alpha x)\,\mathrm{d}_{p,q}x = \frac{1}{\alpha}\exp_{p,q}(\alpha x) + \text{const..} \tag{13b}$$

From the definition of the (p, q)-derivative we can easily derive the (p, q)-integration by parts formula:

$$\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}(f(x)g(x)) = f(qx)\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}g(x) + \left(\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}f(x)\right)g(p^{-1}x) \tag{14a}$$

and

$$\int_0^a f(qx)\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}g(x)\,\mathrm{d}_{p,q}x = f(x)g(x)|_0^a - \int_0^a g(p^{-1}x)\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}f(x)\,\mathrm{d}_{p,q}x. \tag{14b}$$

Note that this result is not unique since we also have

$$\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}(f(x)g(x)) = f(p^{-1}x)\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}g(x) + g(qx)\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}f(x). \tag{15a}$$

Therefore

$$\int_0^a f(p^{-1}x)\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}g(x)\,\mathrm{d}_{p,q}x = f(x)g(x)|_0^a - \int_0^a g(qx)\frac{\mathrm{d}}{\mathrm{d}_{p,q}x}f(x)\,\mathrm{d}_{p,q}x \tag{15b}$$

which can also be obtained by exchanging $q \leftrightarrow p^{-1}$ in (14*b*).

Now let us derive a (p, q)-integration formula, which is the (p, q)-analogue of Euler's formula for $\Gamma(x)$. We first define $-\xi(q, p) < 0$ to be the largest zero of $\exp_{q,p}(x)$. Then we redefine $\exp_{q,p}(x)$ to be

$$\exp_{q,p}(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]_{q,p}!} \tag{16}$$

for $-\xi(q, p) < x$ and zero otherwise.

Using (p, q)-integration by parts (14*b*), we obtain

$$\int_0^{\eta(q,p)} x^n \exp_{q,p}(-qp^{-1}x)\,\mathrm{d}_{q,p}x = (pq^{-1})p^{-n}[n]_{q,p}\int x^{n-1}\exp_{q,p}(-qp^{-1}q^{-1}x)\,\mathrm{d}_{q,p}x \tag{17}$$

where $\eta(q,p)=\xi(q,p)/(qp^{-1})$ and then

$$\int_0^{\eta(q,p)} x^n \exp_{q,p}(-qp^{-1}x)\,\mathrm{d}_{q,p}x = (pq^{-1})^n (p^{-n}[n]_{q,p}q^0)(p^{-(n-1)}[n-1]_{q,p}q^1) \ldots (p^{-1}[1]_{q,p}q^{n-1}) \int_0^{\eta(q,p)} \exp_{q,p}(-qp^{-1}q^{-n}x)\,\mathrm{d}_{q,p}x. \qquad (18)$$

Since

$$\int_0^{\eta(q,p)} \exp_{q,p}(-qp^{-1}q^{-n}x)\,\mathrm{d}_{q,p}x = (pq^{-1})q^n \qquad (19)$$

and

$$[n]_{q,p} = \frac{p^n - q^{-n}}{p-q^{-1}} = (pq^{-1})^{n-1}[n]_{p,q} \qquad (20)$$

we get

$$\int_0^{\eta(q,p)} x^n \exp_{q,p}(-qp^{-1}x)\,\mathrm{d}_{q,p}x = (pq^{-1})[n]_{p,q}!. \qquad (21)$$

This is the (p,q)-analogue of Euler's formula for $\Gamma(x)$.

We now derive a completeness relation for the coherent states of the (p,q)-oscillator by means of (21). Actually, the identity operator can be written as

$$I = \int |z)_{p,q}\;{}_{p,q}(z|\mathrm{d}_{p,q}\mu(z) \qquad (22)$$

where the (p,q)-integration measure

$$\mathrm{d}_{p,q}\mu(z) = \frac{qp^{-1}}{2\pi}\exp_{q,p}(-qp^{-1}|z|^2)\,\mathrm{d}_{q,p}|z|^2\,\mathrm{d}\theta \qquad (23)$$

and the integration over θ (being the argument of z) is a normal one from 0 to 2π, whereas that over $|z|^2$ is a (p,q)-integration from 0 to $\eta(q,p)$. This result follows by

$$\int |z)_{p,q}\;{}_{p,q}(z|\mathrm{d}_{p,q}\mu(z) = \frac{qp^{-1}}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\int \frac{|z|^n|\bar{z}|^m}{\sqrt{[n]_{p,q}![m]_{p,q}!}} \times \exp_{q,p}(-qp^{-1}|z|^2)\,\mathrm{d}_{q,p}|z|^2 \int \mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta\right\}|n\rangle\langle m|$$
$$= \sum_{n=0}^{\infty}\left\{\frac{qp^{-1}}{[n]_{p,q}!}\int |z|^{2n}\exp_{q,p}(-qp^{-1}|z|^2)\,\mathrm{d}_{q,p}|z|^2\right\}|n\rangle\langle n|$$
$$= \sum_{n=0}^{\infty}|n\rangle\langle n| = I. \qquad (24)$$

It is worth noting that the completeness relation in (24) is in agreement with the result in [10] in the limit $p = q$. Furthermore, using the (p, q)-oscillator coherent states $|z\rangle_{p,q}(|z| < \sqrt{\eta(q, p)})$ we can obtain the Bargmann–Fock representation, namely

$$|n\rangle \rightarrow X_n(z) =_{p,q} \langle\bar{z}|n\rangle = \frac{z^n}{\sqrt{[n]_{p,q}!}} \tag{25a}$$

$$|\psi\rangle = \sum_n c_n|n\rangle \rightarrow \psi(z) =_{p,q} \langle\bar{z}|\psi\rangle = \sum_n \frac{c_n z^n}{\sqrt{[n]_{p,q}!}}. \tag{25b}$$

In the Bargmann–Fock space which is the one of analytic functions of complex variable z, the following correspondences can be easily derived

$$A \rightarrow \frac{\mathrm{d}}{\mathrm{d}_{p,q}z} \qquad A^\dagger \rightarrow z \qquad N \rightarrow z\frac{\mathrm{d}}{\mathrm{d}z}. \tag{26}$$

Actually, we have

$$_{p,q}\langle\bar{z}|A|\psi\rangle = \frac{\mathrm{d}}{\mathrm{d}_{p,q}z}\,_{p,q}\langle\bar{z}|\psi\rangle = \frac{\mathrm{d}}{\mathrm{d}_{p,q}z}\psi(z) \tag{27a}$$

$$_{p,q}\langle\bar{z}|A^\dagger|\psi\rangle = z\,_{p,q}\langle\bar{z}|\psi\rangle = z\psi(z) \tag{27b}$$

$$_{p,q}\langle\bar{z}|N|\psi\rangle = z\frac{\mathrm{d}}{\mathrm{d}z}\,_{p,q}\langle\bar{z}|\psi\rangle = z\frac{\mathrm{d}}{\mathrm{d}t}\psi(z). \tag{27c}$$

The inner product can be defined by means of (24) as follows

$$\begin{aligned}\langle\varphi|\psi\rangle &= \int \langle\varphi|\bar{z}\rangle_{p,q}\,_{p,q}\langle\bar{z}|\psi\rangle\,\mathrm{d}_{p,q}\mu(\bar{z}) \\ &= \int \overline{\varphi(z)}\psi(z)\,\mathrm{d}_{p,q}\mu(z)\end{aligned} \tag{28}$$

and with the inner product above we can easily prove

$$(z)^\dagger = \frac{\mathrm{d}}{\mathrm{d}_{p,q}z} \qquad \left(\frac{\mathrm{d}}{\mathrm{d}_{p,q}z}\right)^\dagger = z \qquad \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^\dagger = z\frac{\mathrm{d}}{\mathrm{d}t}. \tag{29}$$

It is shown that the Hermiticity properties $(A^\dagger)^\dagger = A$, $N^\dagger = N$ are entirely retained with respect to the inner product (28). As a final point, let us note that the representations (26), (28) and (29) are the same as in the case of the q-oscillator in the limit $p = q$ [11].

References

[1] Demidov E E, Manin Yu I, Mukhin E E and Zhdanovich D V 1990 *Prog. Theor. Phys. Suppl.* **102** 203
[2] Reshetikhin N Yu 1990 *Lett. Math. Phys.* **20** 331
[3] Sudbery A 1990 *J. Phys. A: Math. Gen.* **23** L697
[4] Chakrabarti R and Jagannathan R 1991 *J. Phys. A: Math. Gen.* **24** L711
[5] Fairlie D B and Zachos C K 1991 *Phys. Lett.* **256B** 43
[6] Smirnov Yu F and Wehrhahn R F 1992 *J. Phys. A: Math. Gen.* **25** 5563
[7] Quesne C 1993 *Phys. Lett.* **174A** 19
[8] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[9] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[10] Gray R W and Nelson C A 1990 *J. Phys. A: Math. Gen.* **23** L945
[11] Quesne C 1991 *Phys. Lett.* **153A** 303